Probability Rules

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Overview

- What’s next?
- Different types of Events
  - Complementary Events
  - Compound Events
  - Union of Events
  - Intersection of Events
- Basic Probability Rules
  - General Additive Rule
  - Additive Rule for Mutually Exclusive Events
  - Multiplicative Law
  - Conditional Probability and Independent Events

Blood Type Problem

- The following data and pie chart is based on the blood types and Rh types of 100 randomly selected people.
- The number of people with each type are given as well as the breakdown of Rh+ and Rh- (in parentheses).
  - Note: Since the numbers sum to 100, it is easy to go back and forth with probabilities.

<table>
<thead>
<tr>
<th>Blood Type</th>
<th>Rh +</th>
<th>Rh -</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>35</td>
<td>5</td>
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<td>Totals</td>
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</tbody>
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Other was to represent this

<table>
<thead>
<tr>
<th>Blood Type</th>
<th>Rh Factor</th>
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<td>A</td>
<td>Rh +</td>
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<tr>
<td>B</td>
<td>Rh +</td>
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<td>Rh +</td>
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- Rh Factor: 86% Rh +, 14% Rh -
Complementary Events

- The Complement of an event A is the event that A does not occur – that is all sample points not in Event A
- Denoted as $A^C$ or as $A'$
- Probability of Complementary Events:
  - $P(A) + P(A^C) = 1.0$
  - $P(A^C) = 1 - P(A)$
- Example using Blood Types
  - $P(\text{Not O Rh–}) = .39 + .01 + .04 + .02 + .08 + .35 + .05 = .94$
  - Or use the complement
    - $P(\text{Not O Rh–}) = 1 - P(\text{O Rh–}) = 1 - .06 = .94$

Compound Events

- Events can be comprised of several events joined together, and these are called COMPOUND EVENTS
- They can be the UNION of several events
- Or the INTERSECTION of several events

Union of Two Events

- The union of two events, A and B is the Event that occurs if either A, B, or both occur on a single performance of the experiment
- We denote the Union as $A \cup B$
- $A \cup B$ consists of all the sample points that belong to A or B or both.
- But, be careful not to count sample points twice
- Example: Union of Type O and Rh+
  - There are 45 type O and 86 Rh+
  - but 39 are both O and Rh+
  - So, there are 92 people who are either O or Rh+
    - $(O \cup Rh+) = 45 + 86 - 39 = 92$
  - And we can draw this out

Venn Diagram of $(O \cup Rh+)$

- $P(O \cup Rh+) =$
- $P(O \cup Rh+) = 45 + 86 - 39 = 92$
Intersection of Two Events \( A \cap B \)

- The **Intersection** of two events, \( A \) and \( B \) is the Event that occurs if both \( A \) and \( B \) occur on a single performance of the experiment.
- We denote the Intersection as \( A \cap B \).
- \( A \cap B \) consists of all the sample points that belong to both \( A \) and \( B \).
- Blood Example of an Intersection
  - Intersection of Type O and Rh+
    - There are 45 type O and 86 Rh+.
    - but 39 are both O and Rh+
    - There are 39 people who are both O and Rh+
- And we can draw this out

Venn Diagram of \((O \cap Rh+)\)

- \( P(O \cap Rh+) = \)
- \( P(O \cap Rh+) = 39 \)

Example of Tossing a Die

- Event \( A \) [Toss an even number]
- Event \( B \) [Toss a number \( <= 3 \)]
- Can you calculate the probability of the complement, union and the intersection of these events?
  - What is \( A^C \)?
  - What is \( A \cup B \)?
  - What is \( A \cap B \)?

What is \( A^C \) for a Roll of a Die?

- \( A = [2, 4, 6] \)
- \( A^C = [1, 3, 5] \)
- the event \( A^C \) is tossing an odd number
- The probability of this event is \( \frac{3}{6} \) or \( \frac{1}{2} \)
  - \( P(A^C) = P(1) + P(3) + P(5) = \frac{3}{6} = \frac{1}{2} \)
- Alternative approach to find \( P(A^C) \):
  - \( P(A^C) = 1 - P(A) = 1 - \{P(2) + P(4) + P(6)\} \)
  - \( P(A^C) = 1 - \frac{3}{6} = 1 - \frac{1}{2} = \frac{1}{2} \)
What is \( A \cup B \) for a Roll of a Die?

- \( A = \{2, 4, 6\} \)
- \( B = \{1, 2, 3\} \)

\[ A \cup B = [2, 4, 6] + [1, 2, 3] = [1, 2, 3, 4, 6] \]

Note that the sample point 2 is common to both events and we don’t count it twice.

Another Way to Solve This

- Find the probability of the complement, and subtract from 1
  - \( P(A \cup B) = 1 - P(A^c) \)
  - \( P(A^c) \) would mean everything that wasn’t in events A or B
  - In this case it is the value of 5
  - And the probability of rolling a five is 1/6
  - \( P(A \cup B) = 1 - 1/6 = 5/6 \)

The probability of this event is 5/6

\[ P(AB) = P(1) + P(2) + P(3) + P(4) + P(6) = 5/6 \]

What is \( (A \cap B) \) for a Roll of a Die?

- \( A = \{2, 4, 6\} \)
- \( B = \{1, 2, 3\} \)

\[ (A \cap B) = [2] \] This is the sample point(s) that are common the A and B

Additive Rule of Probability (General)

- \( P(A \cup B) \)
- \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)

This is called the General Additive Rule of Probability

Let’s see how this applied to the Blood example

- Find the Union of O and Rh+ (i.e., probability of type O or Rh+)
- \( P(O \cup Rh+) = P(O) + P(Rh+) - P(O \cap Rh+) \)
- \( P(O \cup Rh+) = .45 + .86 - .39 = .92 \)

Additive Rule of Probability (in the case of events that are mutual exclusive)

- If events A and B are mutually exclusive, meaning no intersection, then
- Then, \( P(A \cup B) = P(A) + P(B) \)

- Two events are mutually exclusive if when one event occurs in an experiment, the other cannot occur

- Example: Events A and \( A^c \) are mutual exclusive.
- So, \( P(A \cup A^c) = P(A) + P(A^c) \)
Solve these with the Blood Type Data

- $P(A \cup O) = \frac{40}{100} = 0.4$
- $P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6$
- $P(AB \cap Rh+) = \frac{4}{100} = 0.04$

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Solve these with the Blood Type Data

- $P(A \cup O) = P(A \text{ or } O) = P(A) + P(O) = 0.4 + 0.45 = 0.85$
- $P(A^c) = P(\text{not } A) = 1 - P(A) = 1 - 0.4 = 0.6$
- $P(AB \cap Rh+) = \frac{4}{100} = 0.04$

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Are you ready for some more probability rules???

Conditional Probability

- If we have knowledge that affects the outcome of an experiment, the probabilities will be altered
- We call this a Conditional Probability
- Designated as $P(A|B)$
- The probability of Event A is conditioned on the probability of Event B
  - In essence, we change the sample space to Event B
  - This changes the probabilities relative to Event B
- We often use the term "given" when talking about conditional probabilities
  - Given you are registered as a Republican, what is the probability that you support a Democrat candidate
  - Given you are overweight, what is the probability that you have high blood pressure
Conditional Probability

- Suppose we have the roll of a die as our total Sample space
  - S[1, 2, 3, 4, 5, 6]
- Let Event B be rolling an even number
  - B[2, 4, 6]
  - P(B) = P(even number) = P[2, 4, 6] = 3/6 = .5
- Let Event C be a rolling a number less than or equal to 3
  - C[1, 2, 3]
  - P(C) = P(≤ 3) = P[1, 2, 3] = 3/6 = .5
- What if we ask the probability of an even number given the die is less than or equal to 3?
  - P(B|C) = P(even|≤ 3) = P[2|≤ 3] = 1/3
  - Note: it is a 2 out the new or given possible space [1, 2, 3]

Conditional Probability

- The formula of a conditional probability is:
  - \[ P(A | B) = \frac{P(A \cap B)}{P(B)} \]
  - Probability of the intersection of A and B divided by the probability of B
  - It adjusts the probability of the intersection to the reduced sample space of the condition
- Back to the die example: P(B|C)
  - If B = [even number on a die]
  - C = [less than or equal to 3]
  - P(B ∩ C) = P(2) = 1/6 = .1667
  - P(B) = P(1) + P(2) + P(3) = 3/6 = .5
  - P(B|C) = .1667/.5 = .333

Multiplicative Rule

- The Multiplicative Rule shows us the probability of an intersection between two events
- Remember we said a Conditional Probability is determined by the formula
- If we rearrange terms and we can find the formula for the probability of an intersection between A and B
- It shows that the probability of an intersection between two events depends upon the conditional probability between the two events
- This is called the Multiplication Rule of Probability

Special Case of Multiplication Rule if Events are Independent

- In the case of Independence between events A and B
  - Independence means that the Probability of Event A does not depend upon the Probability of B
  - P(A|B) = P(A)
  - With Independence, the formula for probability of an intersection reduces to:
    - P(A ∩ B) = P(A) * P(B)
  - If we can assume independence between events, figuring the probability of the intersection of events becomes much easier
**Multiplication Rule applied to Two flips of a Coin**

- **Experiment**: Flip a coin twice, note the face each time

<table>
<thead>
<tr>
<th>First Flip</th>
<th>Second Flip</th>
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<tbody>
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**Use Multiplicative Rule in cases where we can assume Independence**

- Instead of laying out all the sample points and using the logic that each outcome is equally probable
- We can use the Multiplicative Rule assuming Independence
- We know the first flip is independent to the second flip
- The probability of observing a head in a single flip of a coin is \(\frac{1}{2}\) or .5
- If I can assume independence
  - \(P(\text{two Heads}) = \)
  - \( = P(\text{Head 1st flip}) \times P(\text{Head on the 2nd flip})\)
  - \( = (.5) (.5) = .25\)

**Conditional Probability versus Independence**

- **Conditional Probability** and **Independence** are very important concepts in research
  - If we hypothesize that salary levels differ between men and women, in essence we are saying, “given you are a female, I expect your salary is different.”
  - If we hypothesize that level of response is different between a drug and the treatment group, we are saying, “given you received the drug, your response is higher”
  - We often test conditional probability by comparing the data we observe to a hypothetical model of independence
  - You will see this in future lectures on probabilities using tables

**Look at the Probability Formulas**

- **Probability of a Union**
  \[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

- **Conditional Probability**
  \[ P(A | B) = \frac{P(A \cap B)}{P(B)} \]

- **Probability of an Intersection**
  \[ P(A \cap B) = P(B)P(A | B) \]

**They end up feeling a little circular - you need to know about one to get the other**
Summary

- We covered the notion of Complementary Events
- And Compound Event
  - Unions
  - Intersections
- And then Conditional Probabilities
- And the general Rules of Probabilities
  - Additive Rule
  - Conditional Probability
  - Multiplicative Rule
- Think of the rules as tools that *sometimes* are useful